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REPRESENTATION OF TOLERANCE RELATIONS VALUED IN COMPLETE HEYTING ALGEBRAS AND GRANULATION OF FUZZY INFORMATION

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Abstract. Tolerance relations with an estimate of the intensity in a complete distributive lattice are considered. Given a tolerance relation we construct the space of fuzzy attributes and embed the original space of tolerance in the space of classes of tolerance. The connection of the described construction with the lattices of fuzzy concepts is established. It is proved that any tolerance relation with an estimate in a complete distributive lattice can be represented as a composition of the object-attribute relation and its inverse. The example shows that the above representation may not be unique, even with the minimum requirement. The structure of minimal representations is rather complicated and requires further research.

Keywords: tolerance relation, distributive lattice, knowledge representation, information granules.

The paper deals with tolerance relations valued in the complete distributive lattice (Heyting algebra). We build a set of fuzzy attributes and relate objects with these attributes. Under this construction the initial tolerance relation can be treated as the tolerance relation generated by the relation between objects and attributes. The incentive for writing the paper served researches in two areas of modern science about data. On the one hand, this is a formal concept analysis in its fuzzy version (see [1]). On the other hand, this is a fuzzy clustering and information granules (see [2], [3]).

From the point of view of traditional logic, each concept is characterized by its volume (all objects belonging to the concept) and intent (all attributes shared by those objects). A mathematical model of formal concept was proposed in the 1980s (see [4], [5]). Then the theory of formal concepts was developed both within the framework of traditional mathematics, and in the context of many-valued logics and category theory [6], [7], [8], [9].

The notion of formal concept can be formalized within the framework of the general theory of binary relations. A triple (G, M, I) is called a formal concept, where G is a set of objects, M is a set of attributes, and $I \subseteq G \times M$ is a binary relation. Formal concept generates a Galois connection of subsets of G with subsets of M. Given $A \subseteq G$ let $A' \subseteq M$ be the set of all attributes that are related to all objects in A. Similarly, given $B \subseteq M$ let $B' \subseteq G$ be the set of all objects that are related to all attributes in B. A formal concept is defined as a pair (A, B) such that $A \subseteq G$, $B \subseteq M$ and $A = B'$, $A' = B$ hold. The set A is called the extent while B is called the intent of the concept (A, B).

Formal context (G, M, I) generates a similarity (tolerance) relation $T = I \times I^{-1}$ on G. Objects x and y are related by T if x and y share at least one common attribute. If the intensity of the relationship is estimated in a classical binary scale, any tolerance relation can be represented in this form [10] (see also [11]).

In other words, for an arbitrary tolerance relation R on the set of objects X, one can specify a formal context (X, Y, S) such that $R = S \times S^{-1}$. In the present paper it is shown that this result can be extended to the case when the relations are estimated in a logical scale, which is a complete distributive lattice (Heyting algebra). Thus, the fuzzy tolerance relation is immersed in a context that allows constructing concepts (information granules).

In what follows we suppose that $\boldsymbol{L} = (L, \wedge, \vee, 0, 1)$ is a complete distributive lattice. It will play the role of logical scale.

By an **L**-fuzzy set we mean a fuzzy set with membership values in L. Thus **L**-fuzzy set A is defined by its membership function $m_A: X \to L$. If it is clear which logical scale is involved, we will simply talk about fuzzy subsets. Sometimes instead of $m_A(x)$ we write $A(x)$ for short.

For $\alpha \in L$ we denote by A^α the a-cut of A that is $A^\alpha = \{x \in X \mid A(x) \geq \alpha\}$. If $\beta \geq \alpha$, then $A^\beta \subseteq A^\alpha$. It can be easily seen that $A(x) = \sup\{\alpha \mid x \in A^\alpha\}$. Conversely, let $(A^\alpha)_{\alpha \in L}$ be a family of sets such that $A^\beta \subseteq A^\alpha$ for $\beta \geq \alpha$ and $A^0 = X$. Then there is a fuzzy set A such that A^α is an a-cut for all $\alpha \in L$. The membership function of A is defined by $A(x) = \sup\{\alpha \mid x \in A^\alpha\}$.

Fuzzy sets (with base X) are naturally ordered by cuts:

$A \subseteq B$ iff $A^\alpha \subseteq B^\alpha$ for all $\alpha \in L$.

It can be easily seen that $A \subseteq B$ if and only if $A(x) \leq B(x)$.

Fuzzy relations on X are fuzzy subset of $X \times X$. Fuzzy relation R is defined by its membership function $m_R: X \times X \to L$.

We say that R is

-reflexive if $R(x, x) = 1$ for all $x \in X$,

-symmetric if $R(x, y) = R(y, x)$ for all $x, y \in X$,

-transitive if $R(x, y) \wedge R(y, z) \leq R(x, z)$ for all $x, y, z \in X$.

Given fuzzy relation R it is reflexive, symmetric or transitive if so are all α-cuts R^α considered as crisp relations on X.

We say that R is a tolerance relation if R is reflexive and symmetric. If a tolerance relation R is transitive then R is an equivalence relation.

Let $R: X \times X \to L$ be a tolerance relation. Following [12] we say that fuzzy subset K is a pre-class of tolerance if $K^\alpha \times K^\alpha \subseteq R^\alpha$ for all $\alpha \in L$. Pre-class K is said to be a class of tolerance if K is maximal pre-class with respect to inclusion.

Lemma 1. Any pre-class of tolerance is included in some class of tolerance.

Proof. We show that any chain of pre-classes has upper bound and then apply Zorn's lemma. Let $(K_i)_{i \in I}$ be a chain of pre-classes. We define fuzzy set K putting $K^\alpha = \cup_{i \in I} K_i^\alpha$ for $\alpha \in L$. Obviously K is an upper bound of $(K_i)_{i \in I}$ being considered as a fuzzy set. We have to show that K is a pre-class. Let $x, y \in K^\alpha$. Then we have $x \in K_i^\alpha$, $y \in K_j^\alpha$ for some $i, j \in I$. We may suppose that $K_i^\alpha \subseteq K_j^\alpha$. Then $(x, y) \in K_j^\alpha \times K_j^\alpha \subseteq R^\alpha$.

Lemma 2. For any $a \in X$ there exists a pre-class of tolerance K such that $K(a) = 1$.

Proof. It's enough to put $K^0 = X$ and $K^\alpha = \{a\}$ for $\alpha > 0$.

Using lemma 1 and lemma 2 we get the following theorem..

Theorem 1. For any $a \in X$ there exists a class of tolerance K such that $K(a) = 1$.

Lemma 3. Given $a, b \in X$ we have

$R(a, b) = \sup_K(K(a) \wedge K(b))$,

where K varies over the set of classes of tolerance.

Proof. First, note that there exists a pre-class K' such that $K'(a) \wedge K'(b)$ ³ $R(a, b)$. It suffices to put $K'^\alpha = \{a, b\}$ if $\alpha \leq\varnothing R(a, b)$ and $K'^\alpha = \varnothing$ otherwise. Further, by lemma 1 there exists a class of tolerance K such that $K' \subseteq K$. Then $K(a) \wedge K(b) \geq R(a, b)$ so $\sup_K(K(a) \wedge K(b)) \geq R(\alpha, b)$.

Conversely, Let K be a class of tolerance and $K(a) \wedge K(b) = \alpha$. Then $(a, b) \in K^\alpha \times K^\alpha$. So $(a, b) \in R^\alpha$ where it follows that $R(a, b) \geq \alpha$. Therefore $R(a, b) \geq \sup_K(K(a) \wedge K(b))$.

Denote by Y the set of classes of tolerance. For $K \in Y$ and $a \in X$ we put $S(a, K) = K(a)$. Using the composition of $\boldsymbol{L}$-fuzzy relations we get

$(SS^{-1})(a, b) =$

$p_K(S(a, K) \wedge S^{-1}(K, b)) = \sup_K(K(a) \wedge K(b))$.

In virtue of theorem 1 and lemma 3 we have the following theorem.

Theorem 2. For any $\boldsymbol{L}$-fuzzy tolerance relation R there exists $\boldsymbol{L}$-fuzzy relation S such that

$SS^{-1} = R$.

$\boldsymbol{L}$-fuzzy relation S can be considered as an object-attribute relation. Representation $SS^{-1} = R$ is not unique. It is natural to require S being minimal, i.e. no proper $\boldsymbol{L}$-fuzzy subset S' of S satisfies equation $R = S'S'^{-1}$. In the case when the tolerance relation is transitive, and, therefore, is an equivalence relation, the attribute space and the object-attribute relation are uniquely determined up to natural isomorphism. It should be noted that the attribute space in this case may happen to be a set with $\boldsymbol{L}$-fuzzy equality and not a set with $\boldsymbol{L}$-fuzzy membership (see [13], [14]).

If a tolerance relation is not transitive minimal attribute spaces may differ. We illustrate it by the following example. Let $X = \{x, y, z\}$, and let $L = \{0, 1, u, v\}$ be a Boolean algebra. We define R by $R(x, y) = 1$, $R(y, z) = 1$, and $R(x, z) = 0$. First let Y be a two-element set, $Y = \{a, b\}$. Put $S(x, a) = S(y, a) = S(y, b) = S(z, b) = 1$ and $S(x, b) = S(y, a) = 0$. It can be easily seen that $SS^{-1} = R$ and S is minimal. On the other hand let $Y = \{k, l, m\}$ be a three-element set and $S(y, k) = S(y, l) = S(y, m) = 1$, $S(x, m) = S(z, k) = 0$, $S(x, k) = S(z, l) = u$, $S(x, l) = S(z, m) = v$. It can be easily checked that $SS^{-1} = R$ and S is minimal.

Generally speaking, a set of minimal attribute spaces can have a complex structure. Its study is the subject of further research.

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